Iloczyn Wallisa

dr Bartłomiej Bzdęga

Uniwersytet im. Adama Mickiewicza w Poznaniu

4 listopada 2022

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$$W = \frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdot \frac{6 \cdot 6}{5 \cdot 7} \cdot \frac{8 \cdot 8}{7 \cdot 9} \cdot \ldots = \frac{\pi}{2}$$

Wykład powstał w oparciu o artykuł Johana Wästlunda
An elementary proof of Wallis' product formula for pi.

Iloczyny częściowe

$$\frac{2 \cdot 2}{1 \cdot 3} = \frac{2.66666...}{2}$$

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Iloczyny częściowe

$$\frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 7}\cdot\ldots\cdot\frac{100\cdot 100}{99\cdot 101}=\frac{3.12607\ldots}{2}$$

Iloczyny częściowe

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Iloczyny częściowe

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Granica

$$W_n = \frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 7}\cdot\ldots\cdot\frac{(2n)\cdot(2n)}{(2n-1)\cdot(2n+1)}$$

$$W_n \xrightarrow{n\to\infty} \frac{\pi}{2}$$

$$W = \frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 7}\cdot\frac{8\cdot 8}{7\cdot 9}\cdot\ldots = \frac{\pi}{2}$$

Istnienie granicy

Ciąg $(W) = (W_1, W_2, W_3, \ldots)$ jest rosnący:

$$W_n = \underbrace{\frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\ldots\cdot\frac{(2n-2)\cdot(2n-2)}{(2n-3)\cdot(2n-1)}}_{W_{n-1}}\cdot\underbrace{\frac{(2n)\cdot(2n)}{(2n-1)\cdot(2n+1)}}_{>1},$$

więc $W_n > W_{n-1}$.

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Ciąg (W) jest ograniczony z góry:

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Z powyższych wynika, że istnieje granica

$$W = \lim_{n\to\infty} W_n$$

Szacowanie tempa zbieżności

$$W = W_n \cdot \overbrace{\frac{(2n+2)(2n+2)}{(2n+1)(2n+3)}}^{>1} \cdot \overbrace{\frac{(2n+4)(2n+4)}{(2n+3)(2n+5)}}^{>1} \cdot \overbrace{\frac{(2n+6)(2n+6)}{(2n+5)(2n+7)}}^{>1} \cdot \ldots > W_n$$

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$$W = W_n \cdot \frac{2n+2}{2n+1} \cdot \underbrace{\frac{(2n+2)(2n+4)}{(2n+3)(2n+3)}}_{<1} \cdot \underbrace{\frac{(2n+4)(2n+6)}{(2n+5)(2n+5)}}_{<1} \cdot \ldots < \frac{2n+2}{2n+1} W_n$$

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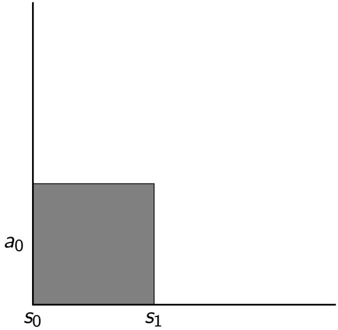
$$\frac{2n+1}{2n+2} W < W_n < W$$

Warstwy prostokątów

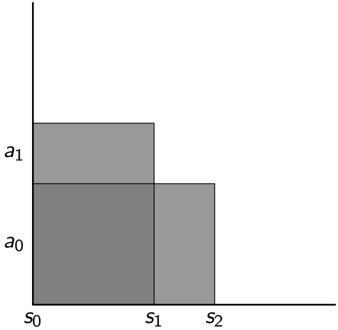


$$a_0 a_0 = 1$$

$$a_0 = 1,$$

$$s_0 = 0, \quad s_1 = 1,$$

Warstwy prostokątów

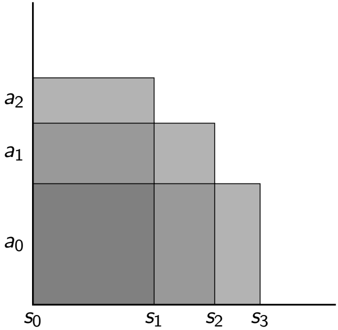


$$a_0 a_0 = 1$$
$$a_0 a_1 + a_1 a_0 = 1$$

$$a_0 = 1, \quad a_1 = \tfrac{1}{2},$$

$$s_0 = 0, \quad s_1 = 1, \quad s_2 = \tfrac{3}{2},$$

Warstwy prostokątów

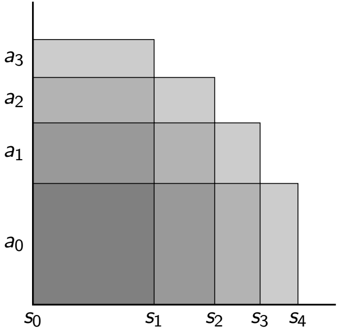


$$a_0 a_0 = 1$$
$$a_0 a_1 + a_1 a_0 = 1$$
$$a_0 a_2 + a_1 a_1 + a_2 a_0 = 1$$

$$a_0 = 1, \quad a_1 = \frac{1}{2}, \quad a_2 = \frac{3}{8},$$

$$s_0 = 0, \quad s_1 = 1, \quad s_2 = \frac{3}{2}, \quad s_3 = \frac{15}{8},$$

Warstwy prostokątów



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$$a_0 = 1, \quad a_1 = \frac{1}{2}, \quad a_2 = \frac{3}{8}, \quad a_3 = \frac{5}{16},$$

$$s_0 = 0, \quad s_1 = 1, \quad s_2 = \frac{3}{2}, \quad s_3 = \frac{15}{8}, \quad s_4 = \frac{35}{16},$$

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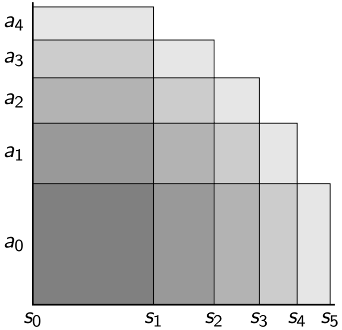

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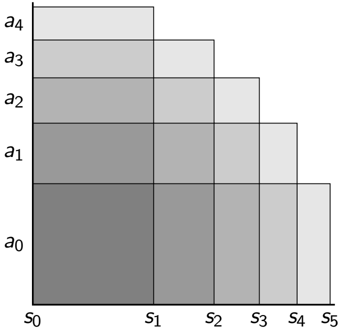
$$\dots$$

$$a_0 = 1, \quad a_1 = \tfrac{1}{2}, \quad a_2 = \tfrac{3}{8},$$
$$a_3 = \tfrac{5}{16}, \quad a_4 = \tfrac{35}{128}, \quad \dots$$

$$s_0 = 0, \quad s_1 = 1, \quad s_2 = \tfrac{3}{2}, \quad s_3 = \tfrac{15}{8}, \quad s_4 = \tfrac{35}{16}, \quad s_5 = \tfrac{315}{128}, \quad \dots$$

Warstwy prostokątów



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Rysunek coraz bardziej przypomina ćwiartkę koła!

Obserwacje wartości s_n

$$s_2 = \frac{3}{2}$$

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Hipoteza:

$$s_n = \frac{3 \cdot 5 \cdot 7 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n-2)}$$

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$$s_n = \frac{3 \cdot 5 \cdot 7 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n-2)}$$

(dowód pomijamy)

Związek s_n z W_n i W

$$s_n = \frac{3 \cdot 5 \cdot 7 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n-2)}$$

$$W_n = \frac{2^2 \cdot 4^2 \cdot 6^2 \cdot \ldots \cdot (2n)^2}{3^2 \cdot 5^2 \cdot \ldots \cdot (2n-1)^2 \cdot (2n+1)}$$

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Na mocy wcześniejszego $\frac{2n+1}{2n+2} W < W_n < W$ otrzymujemy

$$\frac{2n-1}{W} < \frac{4n^2}{W(2n+1)} < \frac{4n^2}{W_n(2n+1)} < \frac{4n^2}{\frac{(2n+1)^2}{2n+2} W} < \frac{2n}{W}$$

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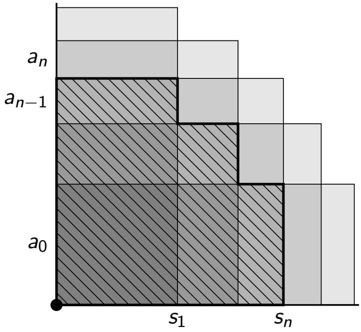
$$\tfrac{2n-1}{W} < s_n^2 < \tfrac{2n}{W}$$

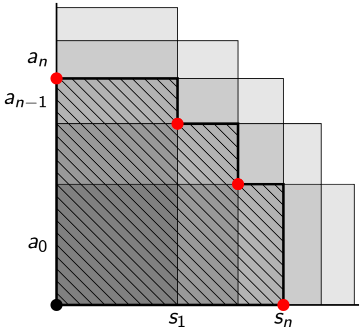
Wielokąt P_n



Wierzchołki:

- $(0, 0)$,

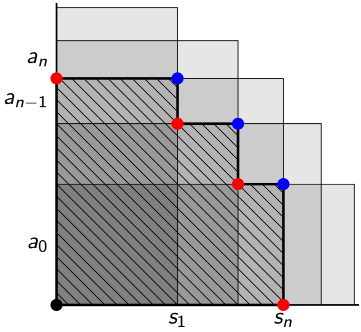
Wielokąt P_n



Wierzchołki:

- $(0, 0)$,
- (s_i, s_j) dla $i + j = n$,

Wielokąt P_n



Wierzchołki:

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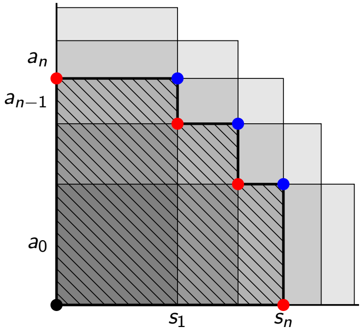
- (s_i, s_j) dla $i + j = n + 1$.

Wielokąt P_n



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Korzystamy z nierówności

$$\frac{2i-1}{W} < s_i^2 < \frac{2i}{W}$$

i analogicznej dla j:

$$\frac{2(n-1)}{W} \leqslant \frac{2i+2j-2}{W} < s_i^2 + s_j^2 < \frac{2i+2j}{W} \leqslant \frac{2(n+1)}{W}$$

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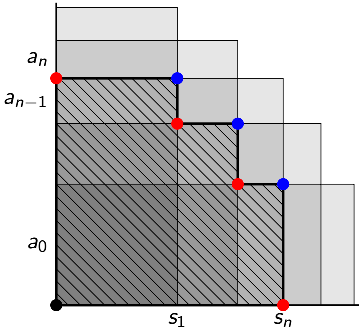

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$$\underbrace{\sqrt{2(n-1)/W}}_{r_{n-1}} < \sqrt{s_i^2+s_j^2} < \underbrace{\sqrt{2(n+1)/W}}_{r_{n+1}}$$

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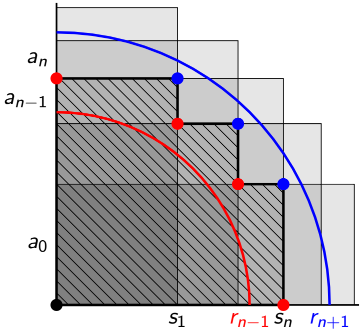

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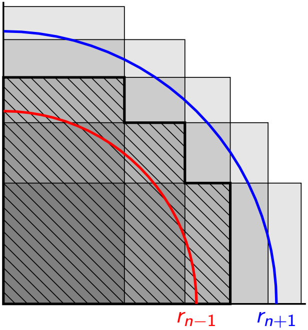
i analogicznej dla j:

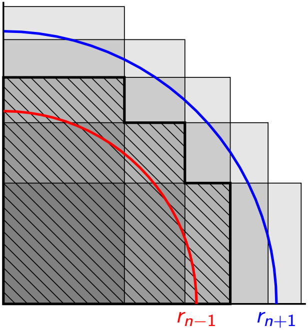
$$\frac{2(n-1)}{W} \leqslant \frac{2i+2j-2}{W} < s_i^2 + s_j^2 < \frac{2i+2j}{W} \leqslant \frac{2(n+1)}{W}$$

$$\underbrace{\sqrt{2(n-1)/W}}_{r_{n-1}} < \sqrt{s_i^2 + s_j^2} < \underbrace{\sqrt{2(n+1)/W}}_{r_{n+1}}$$

Porównanie pól

$$\frac{1}{4}\pi r_{n-1}^2 < [P_n] < \frac{1}{4}\pi r_{n+1}^2$$





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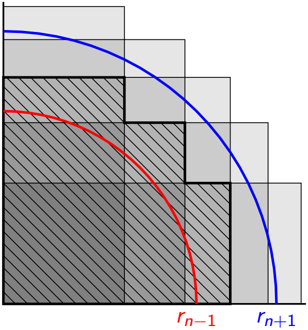

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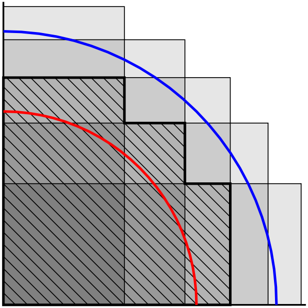

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$$\left(1-\frac{1}{n}\right)\frac{\pi}{2} < W < \left(1+\frac{1}{n}\right)\frac{\pi}{2}$$

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$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

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